

FACTORS OF ALTERNATIVE BINOMIALS SUMS

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ABSTRACT. We confirm several conjectures of Guo, Jouhet and Zeng concerning the factors of alternative binomials sums.

1. INTRODUCTION

It is well-known that

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = (1-1)^n = 0$$

for every positive integer n . However, there are two unfamiliar identities in the same flavor [3, Eqs. (3.81) and (6.6)]:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n} \quad (1.1)$$

and

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{2n}{n} \binom{3n}{n} \quad (1.2)$$

for any $n \geq 1$. Unfortunately, by using asymptotic methods, de Bruijn [1] has showed that no closed form exists for the sum $\sum_{k=0}^n (-1)^k \binom{n}{k}^a$ when $a \geq 4$. Observe that the right sides of (1.1) and (1.2) are both divisible by $\binom{2n}{n}$. Motivated by (1.1) and (1.2), in [2], Calkin established the following interesting congruence:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^r \equiv 0 \pmod{\binom{2n}{n}} \quad (1.3)$$

for any positive integers n and r . Nine years later, Guo, Jouhet and Zeng [4] generalized Calkin's result and showed that for any positive integers $n_1, \dots, n_h, n_{h+1} = n_1$,

$$\sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^h \binom{n_i + n_{i+1}}{n_i + k} \equiv 0 \pmod{\binom{n_1 + n_h}{n_1}} \quad (1.4)$$

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In fact, they proved a q -analogue of (1.4):

$$\sum_{k=-n_1}^{n_1} (-1)^k q^{\binom{k}{2}} \prod_{i=1}^h \begin{bmatrix} n_i + n_{i+1} \\ n_i + k \end{bmatrix}_q \equiv 0 \pmod{\begin{bmatrix} n_1 + n_r \\ n_1 \end{bmatrix}_q}, \quad (1.5)$$

where the above congruence is considered over the polynomials ring $\mathbb{Z}[q]$.

Based on some computer experiments, Guo, Jouhet and Zeng proposed several conjectures on alternative binomial sums:

Conjecture 1.1. *For any positive integers m and n ,*

$$\gcd \left(\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^r : r = m, m+1, \dots \right) = \binom{2n}{n}, \quad (1.6)$$

where $\gcd(a_1, a_2, \dots)$ denotes the greatest common divisor of a_1, a_2, \dots .

Conjecture 1.2. *For any positive integers r, s, t and n ,*

$$\sum_{k=-n}^n (-1)^k \binom{6n}{3n+k}^r \binom{4n}{2n+k}^s \binom{2n}{n+k}^t \equiv 0 \pmod{2 \binom{6n}{n}}, \quad (1.7)$$

$$\sum_{k=-n}^n (-1)^k \binom{6n}{3n+k}^r \binom{4n}{2n+k}^s \binom{2n}{n+k}^t \equiv 0 \pmod{6 \binom{6n}{3n}}. \quad (1.8)$$

Furthermore, if $(r, s, t) \neq (1, 1, 1)$, then

$$\sum_{k=-n}^n (-1)^k \binom{8n}{4n+k}^r \binom{4n}{2n+k}^s \binom{2n}{n+k}^t \equiv 0 \pmod{2 \binom{8n}{3n}}. \quad (1.9)$$

In this paper, we shall confirm these conjectures. For a prime p and an integer n , let $\nu_p(n)$ denote the greatest integer such that $p^{\nu_p(n)} \mid n$. In particular, we set $\nu_p(0) = +\infty$. Let ϕ denote the Euler totient function. Clearly Conjecture 1.1 is implied by the following theorem.

Theorem 1.1. *Suppose that n is a positive integer and r is a positive integer with $r \equiv 2 \pmod{\phi(\binom{2n}{n})}$. Then*

$$\nu_p \left(\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^r \right) = \nu_p \left(\binom{2n}{n} \right)$$

for each prime divisor p of $\binom{2n}{n}$.

For a positive integer n , define

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}.$$

And define the q -binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \prod_{j=1}^k \frac{1 - q^{n+1-j}}{1 - q^j}, & \text{if } n \geq k \geq 1 \\ 1, & \text{if } k = 0, \\ 0, & \text{if } k < 0 \text{ or } n < k. \end{cases}$$

Applying (1.5), it is not difficult (see [4, Theorem 4.7, Corollary 4.10 and Corollary 4.11]) to deduce that

$$\sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \equiv 0 \pmod{\begin{bmatrix} 6n \\ n \end{bmatrix}_q}, \quad (1.10)$$

$$\sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \equiv 0 \pmod{\begin{bmatrix} 6n \\ 3n \end{bmatrix}_q}, \quad (1.11)$$

and

$$\sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 8n \\ 4n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \equiv 0 \pmod{\begin{bmatrix} 8n \\ 3n \end{bmatrix}_q}. \quad (1.12)$$

Now we shall prove that

Theorem 1.2. *Let $\alpha = \nu_2(n)$ and $\beta = \nu_3(n)$. For positive integers r, s, t ,*

$$\begin{bmatrix} 6n \\ n \end{bmatrix}_q^{-1} \sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \equiv 0 \pmod{[2]_{q^{2^\alpha}}} \quad (1.13)$$

and

$$\begin{bmatrix} 6n \\ 3n \end{bmatrix}_q^{-1} \sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \equiv 0 \pmod{[2]_{q^{2^\alpha}} [3]_{q^{2^\alpha}}}. \quad (1.14)$$

Further, we have

$$\begin{aligned} & \begin{bmatrix} 8n \\ 3n \end{bmatrix}_q^{-1} \sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 8n \\ 4n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \\ & \equiv \begin{cases} 0 \pmod{[2]_{q^{2^\alpha}}}, & \text{if } t \geq 2, \\ 0 \pmod{[2]_{q^{2^\alpha+1}}}, & \text{if } s \geq 2, \text{ or } r \geq 2 \text{ and } n \equiv 3 \cdot 2^\alpha \pmod{2^{\alpha+2}}, \\ 0 \pmod{[2]_{q^{2^\alpha+2}}}, & \text{if } r \geq 2 \text{ and } n \equiv 2^\alpha \pmod{2^{\alpha+2}}. \end{cases} \quad (1.15) \end{aligned}$$

Let us explain why Theorem 1.2 implies Conjecture1.2. For example, since $[2]_{q^{2^\alpha}}$ is a primitive polynomial (a polynomial with integral coefficients is called primitive if the greatest common divisor of its coefficients is 1), by (1.13), there exists a polynomial $H(q)$ with integral coefficients such that

$$\sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t = H(q) [2]_{q^{2^\alpha}} \begin{bmatrix} 6n \\ n \end{bmatrix}_q.$$

Thus substituting $q = 1$ in the above equation, we get

$$\sum_{k=-n}^n (-1)^k \binom{6n}{3n+k}^r \binom{4n}{2n+k}^s \binom{2n}{n+k}^t = 2H(1) \binom{6n}{n},$$

that is,

$$\sum_{k=-n}^n (-1)^k \binom{6n}{3n+k}^r \binom{4n}{2n+k}^s \binom{2n}{n+k}^t \equiv 0 \pmod{2 \binom{6n}{n}}.$$

The proofs of Theorems 1.1 and 1.2 will be proposed in Sections 2 and 3.

2. PROOF OF THEOREM 1.1

Suppose that p is an arbitrary prime divisor of $\binom{2n}{n}$ and $\nu_p(\binom{2n}{n}) = \gamma$. Suppose that $r > 2$ be an integer such that

$$r \equiv 2 \pmod{\phi(p^{\gamma+1})}.$$

It is easy to see that $r \geq \gamma + 1$. Then

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^r \equiv \sum_{\substack{0 \leq k \leq 2n \\ p \nmid \binom{2n}{k}}} (-1)^k \binom{2n}{k}^2 \pmod{p^{\gamma+1}}.$$

Thus Theorem 1.1 easily follows from:

Lemma 2.1. *Let p be a prime and n be a positive integer. Then*

$$\nu_p \left(\sum_{\substack{0 \leq k \leq 2n \\ p \nmid \binom{2n}{k}}} (-1)^k \binom{2n}{k}^2 \right) = \nu_p \left(\binom{2n}{n} \right). \quad (2.1)$$

Notice that

$$\sum_{\substack{0 \leq k \leq 2n \\ p \nmid \binom{2n}{k}}} (-1)^k \binom{2n}{k}^2 + \sum_{\substack{0 \leq k \leq 2n \\ p \mid \binom{2n}{k}}} (-1)^k \binom{2n}{k}^2 = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}.$$

So we only need to prove that

Lemma 2.2. *For each $r \geq 1$,*

$$\nu_p \left(\sum_{\substack{0 \leq k \leq 2n \\ p \nmid \binom{2n}{k}}} (-1)^k \binom{2n}{k}^r \right) \geq r - 1 + \nu_p \left(\binom{2n}{n} \right). \quad (2.2)$$

Let

$$\mathcal{D}_{n,k} = \{d \in \mathbb{N} : \lfloor n/d \rfloor > \lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor\},$$

where $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$. Note that $p \mid \binom{2n}{k}$ if and only if the set $\{\beta : p^\beta \in \mathcal{D}_{2n,k}\}$ is non-empty. Letting $h = \lfloor \log_p(2n) \rfloor + 1$, we have

$$\begin{aligned} \sum_{\substack{0 \leq k \leq 2n \\ p \nmid \binom{2n}{k}}} (-1)^k \binom{2n}{k}^2 &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 \sum_{\emptyset \neq I \subseteq \{\alpha : p^\alpha \in \mathcal{D}_{2n,k}\}} (-1)^{|I|-1} \\ &= \sum_{\emptyset \neq I \subseteq \{1,2,\dots,h\}} (-1)^{|I|-1} \sum_{\substack{0 \leq k \leq 2n \\ p^\alpha \in \mathcal{D}_{2n,k}, \forall \alpha \in I}} (-1)^k \binom{2n}{k}^2. \end{aligned}$$

Hence it suffices to show that

Lemma 2.3. For each $\emptyset \neq I \subseteq \{1, \dots, h\}$,

$$\nu_p \left(\sum_{\substack{0 \leq k \leq 2n \\ p^\alpha \in \mathcal{D}_{2n,k}, \forall \alpha \in I}} (-1)^k \binom{2n}{k}^r \right) \geq (r-1)|I| + \nu_p \left(\binom{2n}{n} \right). \quad (2.3)$$

It is not difficult to see that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{d \in \mathcal{D}_{n,k}} \Phi_d(q),$$

where $\Phi_d(q)$ is the d -th cyclotomic polynomial. In particular, we have

$$\Phi_{p^\alpha}(q) = \frac{1 - q^{p^\alpha}}{1 - q^{p^{\alpha-1}}} = [p]_{q^{p^{\alpha-1}}}$$

for every prime p and integer $\alpha \geq 1$. Thus (2.3) is an immediate consequence of the following q -congruence.

Lemma 2.4.

$$\sum_{\substack{0 \leq k \leq 2n \\ p^\alpha \in \mathcal{D}_{2n,k}, \forall \alpha \in I}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^r \equiv 0 \pmod{\prod_{\alpha \in I} \Phi_{p^\alpha}(q)^r \prod_{\substack{\beta \notin I \\ p^\beta \in \mathcal{D}_{2n,n}}} \Phi_{p^\beta}(q)}. \quad (2.4)$$

Proof. We need a q -analogue of well-known Lucas' congruence (cf. [5]):

$$\begin{bmatrix} x_1 d + x_2 \\ y_1 d + y_2 \end{bmatrix}_q \equiv \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}_q \pmod{\Phi_d(q)} \quad (2.5)$$

for every $d \geq 2$, where $0 \leq x_2, y_2 < d$.

For any β with $\beta \notin I$ and $p^\beta \in \mathcal{D}_{2n,n}$, write $n = n_1 p^\beta + n_2$ with $0 \leq n_2 < p^\beta$. Since $p^\beta \in \mathcal{D}_{2n,n}$, we have $2n_2 \geq p^\beta$. For any $k = k_1 p^\beta + k_2$ with $0 \leq k_2 < p^\beta$, by (2.5),

$$\begin{bmatrix} 2n \\ k \end{bmatrix}_q \equiv \begin{pmatrix} 2n_1 + 1 \\ k_1 \end{pmatrix} \begin{bmatrix} 2n_2 - p^\beta \\ k_2 \end{bmatrix}_q \pmod{\Phi_{p^\beta}(q)}.$$

Hence

$$\begin{bmatrix} 2n \\ k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{p^\beta}(q)}.$$

provided that $2n_2 - p^\beta < k_2$.

Suppose that $2n_2 - p^\beta \geq k_2$. Assume that $I = \{\alpha_1, \alpha_2, \dots, \alpha_u\}$ with

$$\alpha_1 < \alpha_2 < \dots < \alpha_v < \beta < \alpha_{v+1} < \dots < \alpha_u.$$

When $1 \leq j \leq v$, we have

$$\begin{aligned} & \left\lfloor \frac{2n}{p^{\alpha_j}} \right\rfloor - \left\lfloor \frac{k}{p^{\alpha_j}} \right\rfloor - \left\lfloor \frac{2n-k}{p^{\alpha_j}} \right\rfloor \\ &= \left\lfloor \frac{(2n_1 + 1)p^\beta + 2n_2 - p^\beta}{p^{\alpha_j}} \right\rfloor - \left\lfloor \frac{k_1 p^\beta + k_2}{p^{\alpha_j}} \right\rfloor - \left\lfloor \frac{(2n_1 + 1 - k_1)p^\beta + 2n_2 - p^\beta - k_2}{p^{\alpha_j}} \right\rfloor \\ &= \left\lfloor \frac{2n_2 - p^\beta}{p^{\alpha_j}} \right\rfloor - \left\lfloor \frac{k_2}{p^{\alpha_j}} \right\rfloor - \left\lfloor \frac{2n_2 - p^\beta - k_2}{p^{\alpha_j}} \right\rfloor. \end{aligned}$$

It follows that $p^{\alpha_j} \in \mathcal{D}_{2n,k}$ if and only if $p^{\alpha_j} \in \mathcal{D}_{2n_2-p^\beta, k_2}$ for $1 \leq j \leq v$. Similarly,

$$\begin{aligned} & \left\lfloor \frac{(2n_1+1)p^\beta + 2n_2 - p^\beta}{p^{\alpha_j}} \right\rfloor - \left\lfloor \frac{k_1 p^\beta + k_2}{p^{\alpha_j}} \right\rfloor - \left\lfloor \frac{(2n_1+1-k_1)p^\beta + 2n_2 - p^\beta - k_2}{p^{\alpha_j}} \right\rfloor \\ &= \left\lfloor \frac{2n_1+1}{p^{\alpha_j-\beta}} \right\rfloor - \left\lfloor \frac{k_1}{p^{\alpha_j-\beta}} \right\rfloor - \left\lfloor \frac{2n_1+1-k_1}{p^{\alpha_j-\beta}} \right\rfloor \end{aligned}$$

provided that $\alpha_j > \beta$. Therefore $p^{\alpha_j} \in \mathcal{D}_{2n,k}$ if and only if $p^{\alpha_j-\beta} \in \mathcal{D}_{2n_1+1, k_1}$ for $v+1 \leq j \leq u$. Thus

$$\begin{aligned} & \sum_{\substack{0 \leq k \leq 2n \\ p^\alpha \in \mathcal{D}_{2n,k}, \forall \alpha \in I}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \\ & \equiv \sum_{\substack{0 \leq k_1 \leq 2n_1+1 \\ p^{\alpha_j-\beta} \in \mathcal{D}_{2n_1+1, k_1}, \\ \forall j \in \{v+1, \dots, u\}}} (-1)^{k_1 p^\beta} q^{\binom{k_1 p^\beta}{2}} \binom{2n_1+1}{k_1}^r \cdot \sum_{\substack{0 \leq k_2 \leq 2n_2-p^\beta \\ p^{\alpha_j} \in \mathcal{D}_{2n_2-p^\beta, k_2}, \\ \forall j \in \{1, \dots, v\}}} (-1)^{k_2} q^{\binom{k_2}{2}} \begin{bmatrix} 2n_2-p^\beta \\ k_2 \end{bmatrix}_q^r \\ & \pmod{\Phi_{p^\beta}(q)}, \end{aligned}$$

by noting that

$$q^{\binom{k}{2}} = q^{\binom{k_1 p^\beta + k_2}{2}} = q^{\binom{k_1 p^\beta}{2} + \binom{k_2}{2} + k_1 k_2 p^\beta} \equiv q^{\binom{k_1 p^\beta}{2} + \binom{k_2}{2}} \pmod{\Phi_{p^\beta}(q)}.$$

If p is an odd prime, then

$$q^{\binom{k_1 p^\beta}{2}} = (q^{p^\beta})^{\frac{k_1(k_1 p^\beta - 1)}{2}} \equiv 1 \pmod{\Phi_{p^\beta}(q)}.$$

And if $p = 2$, then we have

$$q^{\binom{k_1 2^\beta}{2}} = (q^{2^{\beta-1}})^{k_1(k_1 2^\beta - 1)} \equiv (-1)^{k_1} \pmod{\Phi_{2^\beta}(q)}$$

since $1 + q^{2^{\beta-1}} = [2]_{q^{2^{\beta-1}}} = \Phi_{2^\beta}(q)$. Notice that $\mathcal{D}_{2n_1+1, k_1} = \mathcal{D}_{2n_1+1, 2n_1+1-k_1}$. We have

$$\begin{aligned} & \sum_{\substack{0 \leq k_1 \leq 2n_1+1 \\ p^{\alpha_j-\beta} \in \mathcal{D}_{2n_1+1, k_1}, \\ \forall j \in \{v+1, \dots, u\}}} (-1)^{k_1 p^\beta} q^{\binom{k_1 p^\beta}{2}} \binom{2n_1+1}{k_1}^r \\ & \equiv \frac{1}{2} \sum_{\substack{0 \leq k_1 \leq 2n_1+1 \\ p^{\alpha_j-\beta} \in \mathcal{D}_{2n_1+1, k_1}, \\ \forall j \in \{v+1, \dots, u\}}} ((-1)^{k_1} + (-1)^{2n_1+1-k_1}) \binom{2n_1+1}{k_1}^r = 0 \pmod{\Phi_{p^\beta}(q)}. \end{aligned}$$

Finally, clearly

$$\sum_{\substack{0 \leq k \leq 2n \\ p^\alpha \in \mathcal{D}_{2n,k}, \forall \alpha \in I}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^r \equiv 0 \pmod{\Phi_{p^\alpha}(q)^r}$$

for any $\alpha \in I$. □

3. PROOF OF THEOREM 1.2

Recalling that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{d \in \mathcal{D}_{n,k}} \Phi_d(q)$ and $\Phi_{p^\alpha}(q) = [p]_{q^{p^\alpha-1}}$. Let $\alpha = \nu_2(n)$. For any k with $\nu_2(k) \neq \alpha$, since

$$2n \equiv 0 \pmod{2^{\alpha+1}} \quad \text{and} \quad n+k \not\equiv 0 \pmod{2^{\alpha+1}},$$

we have

$$\begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q)}.$$

Similarly,

$$\begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q)}.$$

Hence

$$\sum_{\substack{-n \leq k \leq n \\ \nu_2(k) \neq \alpha}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q)^2}. \quad (3.1)$$

On the other hand, obviously

$$\begin{aligned} & \sum_{\substack{-n \leq k \leq n \\ \nu_2(k) = \alpha}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \\ &= \sum_{\substack{k > 0 \\ \nu_2(k) = \alpha}} (-1)^k q^{\binom{k}{2}} (1+q^k) \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t. \end{aligned}$$

For any k with $\nu_2(k) = \alpha$, we have

$$4n \equiv 0 \pmod{2^{\alpha+1}} \quad \text{and} \quad 2n+k \equiv 2^\alpha \pmod{2^{\alpha+1}},$$

whence

$$\begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q)}.$$

And $1+q^k$ is divisible by $1+q^{2^\alpha} = \Phi_{2^{\alpha+1}}(q)$, since $k/2^\alpha$ is odd. Thus

$$\sum_{\substack{-n \leq k \leq n \\ \nu_2(k) = \alpha}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q)^2}. \quad (3.2)$$

Combining (3.1) and (3.2), we have

$$\sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q)^2}. \quad (3.3)$$

And by (3.3) and (1.10), we conclude that

$$\sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q) \begin{bmatrix} 6n \\ n \end{bmatrix}_q},$$

since $\Phi_{2^{\alpha+1}}(q)^2 \nmid \begin{bmatrix} 6n \\ n \end{bmatrix}_q$.

Let $\beta = \nu_3(n)$. If $\nu_3(k) \leq \beta$, then

$$6n \equiv 3n \equiv 0 \pmod{3^{\beta+1}} \quad \text{and} \quad 3n + k \not\equiv 0 \pmod{3^{\beta+1}},$$

whence

$$\begin{bmatrix} 6n \\ 3n + k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{3^{\beta+1}}(q)}.$$

Suppose that $\nu_3(k) > \beta$. If $n \equiv 3^\beta \pmod{3^{\beta+1}}$. Then

$$4n \equiv 3^\beta \pmod{3^{\beta+1}} \quad \text{and} \quad 2n + k \equiv 2 \cdot 3^\beta \pmod{3^{\beta+1}}.$$

Thus

$$\begin{bmatrix} 4n \\ 2n + k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{3^{\beta+1}}(q)}.$$

And if $n \equiv 2 \cdot 3^\beta \pmod{3^{\beta+1}}$, then

$$2n \equiv 3^\beta \pmod{3^{\beta+1}} \quad \text{and} \quad n + k \equiv 2 \cdot 3^\beta \pmod{3^{\beta+1}},$$

whence

$$\begin{bmatrix} 2n \\ n + k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{3^{\beta+1}}(q)}.$$

This concludes that

$$\sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 6n \\ 3n + k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n + k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n + k \end{bmatrix}_q^t \equiv 0 \pmod{\Phi_{3^{\beta+1}}(q)}. \quad (3.4)$$

Since $6n \equiv 3n \equiv 0 \pmod{3^{\beta+1}}$, $3^{\beta+1} \notin \mathcal{D}_{6n, 3n}$, i.e., $\Phi_{3^{\beta+1}}(q) \nmid \begin{bmatrix} 6n \\ 3n \end{bmatrix}_q$. Thus combining (3.3), (3.4) and (1.11), we get (1.14).

Finally, let us turn to (1.9). Suppose that $\nu_2(n) = \alpha$. Since $(r, s, t) \neq (1, 1, 1)$, we may consider the following three cases:

Case 1: $t \geq 2$. If $\nu_2(k) \neq \alpha$, then

$$2n \equiv 0 \pmod{2^{\alpha+1}} \quad \text{and} \quad n + k \not\equiv 0 \pmod{2^{\alpha+1}},$$

whence

$$\begin{bmatrix} 2n \\ n + k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q)}.$$

And if $\nu_2(k) = \alpha$, then

$$8n \equiv 4n \equiv 0 \pmod{2^{\alpha+1}} \quad \text{and} \quad 4n + k \equiv 2n + k \equiv 2^\alpha \pmod{2^{\alpha+1}}.$$

So

$$\begin{bmatrix} 8n \\ 4n + k \end{bmatrix}_q \equiv \begin{bmatrix} 4n \\ 2n + k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q)}.$$

Hence

$$\sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 8n \\ 4n + k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n + k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n + k \end{bmatrix}_q^t \equiv 0 \pmod{\Phi_{2^{\alpha+1}}(q)^2}. \quad (3.5)$$

Case 2: $s \geq 2$. If $\nu_2(k) \neq \alpha + 1$, then

$$4n \equiv 0 \pmod{2^{\alpha+2}} \quad \text{and} \quad 2n + k \not\equiv 0 \pmod{2^{\alpha+2}},$$

whence

$$\begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+2}}(q)}.$$

Assume that $\nu_2(k) = \alpha + 1$. Then

$$8n \equiv 0 \pmod{2^{\alpha+2}} \quad \text{and} \quad 4n+k \equiv 2^{\alpha+1} \pmod{2^{\alpha+2}}.$$

It follows that

$$\begin{bmatrix} 8n \\ 4n+k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+2}}(q)}.$$

And $\Phi_{2^{\alpha+2}}(q) = 1 + q^{2^{\alpha+1}}$ divides $1 + q^k$ since $k/2^{\alpha+1}$ is odd. Thus

$$\begin{aligned} & \sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 8n \\ 4n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \\ & \equiv \sum_{\substack{-n \leq k \leq n \\ \nu_2(k)=\alpha+1}} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 8n \\ 4n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \\ & = \sum_{\substack{0 < k \leq n \\ \nu_2(k)=\alpha+1}} (-1)^k q^{\binom{k}{2}} (1 + q^k) \begin{bmatrix} 8n \\ 4n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \\ & \equiv 0 \pmod{\Phi_{2^{\alpha+2}}(q)^2}. \end{aligned} \tag{3.6}$$

Case 3: $r \geq 2$. We consider two subcases:

(i) $n \equiv 2^\alpha \pmod{2^{\alpha+2}}$. For any k with $\nu_2(k) \neq \alpha + 2$, we have

$$8n \equiv 0 \pmod{2^{\alpha+3}} \quad \text{and} \quad 4n+k \not\equiv 0 \pmod{2^{\alpha+3}}.$$

So

$$\begin{bmatrix} 8n \\ 4n+k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+3}}(q)}.$$

And for any k with $\nu_2(k) = \alpha + 2$, we have

$$4n \equiv 2^{\alpha+2} \pmod{2^{\alpha+3}} \quad \text{and} \quad 2n+k \equiv 2^{\alpha+2} + 2^{\alpha+1} \pmod{2^{\alpha+3}}.$$

Then

$$\begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q \equiv 1 + q^k \equiv 0 \pmod{\Phi_{2^{\alpha+3}}(q)}.$$

Thus

$$\begin{aligned} & \sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 8n \\ 4n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \\ & \equiv \sum_{\substack{0 < k \leq n \\ \nu_2(k)=\alpha+2}} (-1)^k q^{\binom{k}{2}} (1 + q^k) \begin{bmatrix} 8n \\ 4n+k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n+k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^t \\ & \equiv 0 \pmod{\Phi_{2^{\alpha+3}}(q)^2}. \end{aligned} \tag{3.7}$$

(ii) $n \equiv 3 \cdot 2^\alpha \pmod{2^{\alpha+2}}$. For any k with $\nu_2(k) < \alpha + 2$, we have

$$8n \equiv 4n \equiv 0 \pmod{2^{\alpha+2}} \quad \text{and} \quad 4n + k \not\equiv 0 \pmod{2^{\alpha+2}},$$

whence

$$\begin{bmatrix} 8n \\ 4n + k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+2}}(q)}.$$

If $\nu_2(k) \geq \alpha + 2$, then

$$4n \equiv 0 \pmod{2^{\alpha+2}}, \quad 2n \equiv 2n + k \equiv 2^{\alpha+1} \pmod{2^{\alpha+2}}, \quad n + k \equiv 3 \cdot 2^\alpha \pmod{2^{\alpha+2}}.$$

Hence

$$\begin{bmatrix} 4n \\ 2n + k \end{bmatrix}_q \equiv \begin{bmatrix} 2n \\ n + k \end{bmatrix}_q \equiv 0 \pmod{\Phi_{2^{\alpha+2}}(q)}$$

and

$$\sum_{k=-n}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 8n \\ 4n + k \end{bmatrix}_q^r \begin{bmatrix} 4n \\ 2n + k \end{bmatrix}_q^s \begin{bmatrix} 2n \\ n + k \end{bmatrix}_q^t \equiv 0 \pmod{\Phi_{2^{\alpha+2}}(q)^2}. \quad (3.8)$$

From (3.5)-(3.8) and (1.12), (1.15) is concluded.

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